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G.R.I.M. (Department of Mathematics, University of Palermo, Italy)

SECONDARY SCHOOL STUDENTS’ UNDERSTANDING OF FUNCTION VIA EXPLORATORY AND INVESTIGATIVE TASKS¹

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Abstract: *This paper aims to present secondary school students’ understanding of function, in an exploratory and investigative approach, and to identify the students’ difficulties concerning that concept. It begins with a presentation of the ways in which researchers describe the symbol sense, the understanding of the concept of function, and the approaches with exploratory and investigative tasks. Subsequently, it presents the qualitative methodology used in the study, and finally its results. The students were able to manipulate the symbols, and operate with them, but this was not sufficient for the structural understanding of function. Also, this study underscores that an exploratory and investigative approach is a rich way to learn functions.*

Keywords: Function; symbol sense; symbol manipulation; exploratory task; investigative task.

Introduction

Function is one of the basic mathematical concepts. The students face many difficulties when they attempt to understand it and when they need to use the chain of symbols that are connected with it. It is a crucial subject of the portuguese curriculum of the secondary school (10th, 11th, and 12th grades). This paper presents the results of a study developed with students of a grade 11 class, in the topic *Functions II*. It presents the secondary school students’ understanding of function, in an exploratory and investigative approach, and identifies the students’ difficulties concerning that concept.

Symbol sense

Algebraic thought, for Ponte (2005), includes the capacity to work with algebraic calculation and functions, and the capacity to deal with many other mathematical structures, as well as the capacity to use the symbols in the interpretation and resolution of mathematical problems in other domains. It still includes the symbol manipulation and the symbol sense. This, for Arcavi (2005), consists of the capacity to interpret and to use, in a creative way, mathematical symbols in the description of situations and in problem solving, such as the choice of symbols, the flexible manipulation skills, and the symbols in context. On the other hand, Rojano (2002) reminds us of the importance of establishing how the meaning variation of mathematical symbols during the transition from arithmetic to algebra represents an obstacle in the subject’s evolution toward the acquisition of algebraic language. These differences in meaning of the same symbols and symbol chains present serious difficulties for secondary school students in the learning of algebra, challenging the old idea that algebra could be conceived, for teaching purposes, as “an extension of arithmetic” (p. 145). Rojano (2002) says, additionally, that the majority of students in secondary school are not able to connect by themselves the knowledge domains that constitute manipulative algebra on the one hand and instrumental algebra for problem solving on the other.

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The transition from the particular to the general also has been discussed, namely the haste to symbolization, during the accomplishment of the tasks of generalization in the classroom. Inadequately, teachers usually have an apparent tendency in teaching to abbreviate the process, and as a result they do not provide students with the opportunity to formulate the algebraic equation for the stated problem.

For Chazan and Yerushalmy (2003), school algebra might be thought of as cut and dried. To these authors, symbol chains involved in the problems of algebra evade a simple classification, which implies that the students cannot know the concrete methods they may choose to solve the specific problems, or that they may choose incorrectly a method to solve the problems. This means that it is necessary to develop a curriculum that does more than teaching specific methods to solve certain types of problems, where “the instruction should connect with students’ experience and build on the resources and strengths present in the conceptions they bring to school” (p. 133).

In this study we adopt the concept of symbol sense given by Arcavi (2005). We also think that it is necessary that students obtain a feeling about the most appropriate methods to work with the symbol chains and that they should appreciate the meanings of those methods.

The understanding of the concept of function

The concept of function is one of the fundamental mathematical concepts. It is extraordinary in the diversities of its interpretations and representations. However, students face many difficulties when they try to understand it. Chazan *et al.* (2003) say that usually functions are conceptualized as a special type of relation. In fact, every linear equation can be written as an equivalent equation which is also a linear function in one variable. To create a graph of a linear equation with two variables it may be useful to write it as a linear function with one variable – Chazan *et al.* (2003) emphasise the connections between the graphs and the expressions as eventual benefits to the understanding of the existing equivalences and the differences. In this sense, they pose questions such as: How to develop the curriculum to help the students promote a sense of the different types of symbols and of the various uses of the concepts such as variable, equals sign, and the system of cartesian coordinates? Does it matter how students learn these notions? Does the introduction sequence matter? With these questions they highlight how important are the role of the symbols and of the different representations of the functions and also the didactical approaches.

To Sajka (2003), one of the students’ difficulties in understanding the concept of function stems from its dual nature. In fact, and in accordance with Sfard (1991), the function can be understood in two essentially different ways: i) structurally – as an object; and ii) operationally – as a process. In the first, the function is a set of ordered pairs, and in the operational way it is a computational process or well defined method for getting from one system to another. These two ways of understanding functions, although apparently ruling out one another, however, should complete each other and constitute a coherent unity – like two sides of the same coin (p. 230). For example, $f(x)=2x+3$ tells us two things at the same time: how to calculate the value of the function for particular arguments (evoking the process) and it encapsulates the whole concept of function for any given argument (thus presenting the object). So, we can say that $f(x)$ represents both the name of a function and the value of the function f . And, in the context of functions, when we write y , sometimes we are referring to a certain value of the function; at

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other times we are referring to the ordinate of a certain point in the coordinates system, and yet in other times we are referring to an argument. The interpretation depends on the context, which can confuse a non-advanced student. This notation of function is ambiguous and promotes some difficulties among students. For Sajka (2003), the causes of students' symbols difficulties also depend on the contexts in which the symbols are worked in mathematics classes, and on the teachers' limited choices of mathematical tasks. For this author, the concept of function is often linked to the concept of formula, and sometimes the students connect the concept of function to the graphing process, where a formula is necessary to draw it.

So it is very important for teachers' professional practice to take into account the existence of ambiguous notation, like the function one. Also, it is crucial the role of the mathematical experience in mathematics classes, where the teacher must propose to the students mathematical tasks with open choices.

The mathematics class with exploratory and investigative tasks

To learn mathematics is, for us, to understand its nature. Goldenberg (1999) says that it is very important to develop the students' mathematical activity because one of the aims of mathematics education must be to make students learn how mathematicians discover mathematical methods and facts. For this reason, it is fundamental that students spend time with exploratory and investigative mathematical tasks. The aim is for the students to learn how to be astute researchers, and for that reason, it is necessary that they do investigation. This idea is clear in many curricular orientations in several countries. For NCTM (2000), the key to the promotion of the students' performance in a certain domain, like school algebra, is not the creation of an ever more elaborate and finely-tuned set of procedures, but rather by changing the nature of instruction. One cannot ignore students' conceptions and it is necessary to confront students' misconceptions. Although the focus of learning would not be exclusively exploratory and investigative tasks (there are others, such as exercises), these activities can promote the students' engagement in creating and discovering genuine mathematical processes (Pereira, 2004; Ponte *et al*, 1998; Teixeira, 2005).

The teacher's role in promoting the students' mathematical activity is crucial. The students' interest will be stimulated by the mathematical tasks selected by the teacher, and by the situations and contexts that the teacher promotes in the class, as well as by their capacity to develop and to lead the students' activity with success. It will be the mathematical tasks and situations that give the opportunity to the students to develop their own algebraic thinking. Also, to get a good integration of exploratory and investigative tasks the teacher needs not only to mobilize theories and techniques but also to mobilize his conceptions, feelings and practical knowledge (Saraiva, 2001).

Methodology

A three month-long study (February – May, 2007) was conducted in a secondary school in Covilhã, Portugal: The topic *Functions II* course (K11 level), which made use of graphing calculator, took place in a mathematics class with 24 students – 16/18 years old; laborious; and, in general, they liked mathematics, though some of them had difficulties in mathematics subject. The pedagogical proposal elaborated is synthesized in the next figure (figure 1). In accordance with the pedagogical proposal previously elaborated, taking into account the curricular requirements and the problem of the study, the topic *Functions II* was developed with emphasis on the

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resolution of exploratory and investigative mathematical tasks (some of them with the use of the graphic calculator), along with sessions of verbal exposition of the programmatic contents, by the teacher, and of the resolution of problems and exercises. Many times the students worked in groups, solving the exploratory and investigative mathematical tasks. Then, they wrote a report of their work and after that there was a time to discuss the results in the whole class. The teacher of the class is the second author of this paper.

The pedagogical proposal	
<i>Task</i>	<i>Objectives</i>
Questionnaire 1 (at the beginning of the study) – Q _I	To identify students' previous knowledge on functions
Transformations of rational functions	To explore the influence of the parameters b , d and a , respectively, using the graphic calculator, in the functions family of the type $y = \frac{b}{x}$ ($b \neq 0$), $y = \frac{1}{x+d}$, and $y = a + \frac{1}{x}$
Teacher' Exposition/Problem solving/exercises from the text book and from the worksheets built by the teacher	To give some mathematical information (by the teacher); to drill and practice (the students)
Operations with functions	To explore and to investigate situations to introduce the addition, difference and product operations with functions (analytically, graphically, and with tables)
Teacher' Exposition/Problem solving/exercises from the text book and from the worksheets built by the teacher	To give some mathematical information (by the teacher); to drill and practice (the students)
Questionnaire 2 (at the end of the study) – Q _F	To identify students' final knowledge on functions
Figure 1 – Synthesis of the pedagogical proposal.	

In this study a qualitative research methodology was followed. The data were collected through i) two questionnaires (Q_I and Q_F) that the students answered – the first one at the beginning of the course and the other one at the end of the teaching of the topic *Functions II*; ii) the oral comments of the students while working in groups and in the sessions of group discussion, registered by the teacher/researcher in her diary; and iii) the students' resolutions of some tasks. The analysis started at the beginning of the study, however, a more intense analysis took place after the collection of all the data. Initially the data were grouped by instrument of collection (diary, questionnaire, etc.), and later by category, with some questioning (*function* – definition, representations, connections, symbols, difficulties; *other aspects of algebra* – algebraic manipulation, symbols, difficulties; *resolutions and original strategies of the students* – creativity, relations). By this way, three groups of data were constructed. Afterwards, each group was ana-

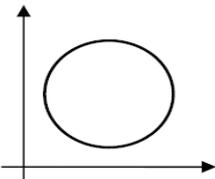
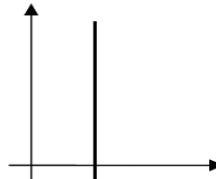
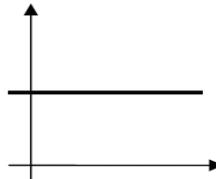
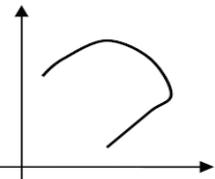
lysed, having in account the problem of the study and the literature review. Finally, the conclusions were carried out.

Results of the study

Function

In the first lesson of the topic *Functions II* the questionnaire Q_1 was distributed to the students. The main purpose was to identify the students' previous knowledge of functions. From analysis of their answers, some aspects deserved special attention. In the first phase students were asked to identify, among the given graphs, the one(s) that represented a function and, afterwards, to say in their own words, what they understood by the term 'function' (figure 2).

1. Choose the graphic(s) that represents(represent) a function. Justify.

(A)  (B)  (C)  (D) 

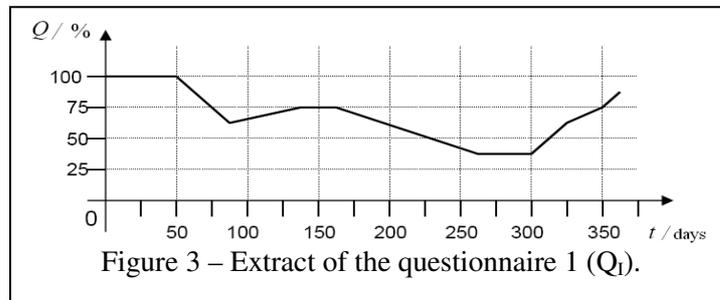
2. Say, using own words, what is a function.

Figure 2 - Extract of the questionnaire 1 (Q_1).

Only seven of the twenty-four students identified, correctly, option (C); the majority (sixteen students) chose options (B) and (C); only one student opted for graph (A).

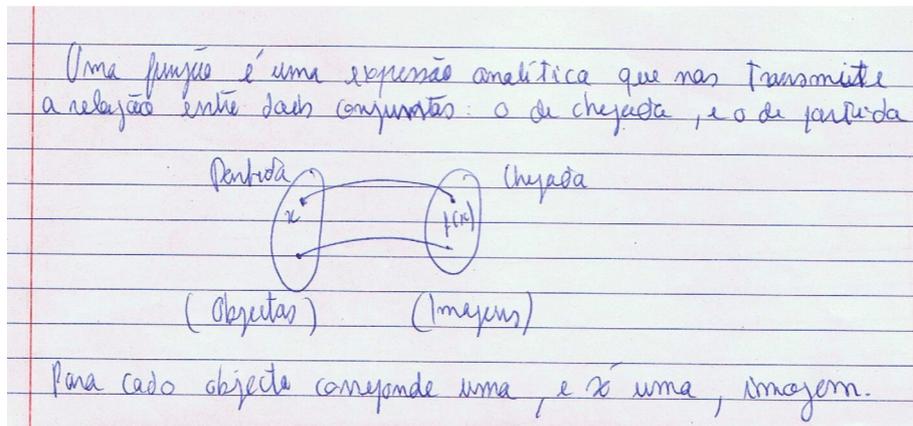
From the data we can say that some students do not successfully connect the definition of function they write with their choice of graph that represents the function. For example, the definition “In a function, to each object corresponds one and only one image” is associated to the choice of both options – (B) and (C). Also, some students presented the statement “there are no repeated points” and still chose graph (C). It seems that, for them, the definition was only words.

The relation of dependence between the variables was one of the aspects highlighted by definitions given by the students, as we can see in the next answer: “In a function there is a dependent variable (y) and an independent variable (x)”. In another answer, this idea was expanded: “ y changes in the function of x and each x gives one and only one value for y ”. However this relation of dependence seems to be confined to the symbols y and x . In fact, in the answers to question 4 of the questionnaire Q_1 (figure 3), where a graph was given showing the variation of the amount of water in a dam throughout one year, as a function of time, measured in days, and students were asked to identify the dependent and the independent variables, some students attributed the symbol x to the number of days (t), and the symbol y to the existing water (Q).

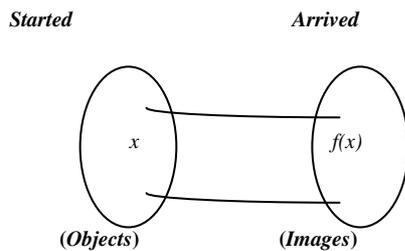


Also, for the students, the variable concept has a *static* meaning. They did not convey an idea of variation, as we can see in the answers to this question, where the students write “days” (not “number of days”) as the independent variable, and “water” (not “amount of water”) as the dependent variable.

In the questionnaire 2 (Q_F) after the study, and concerning to the question “Say, using own words, what is a function. Give two examples of function“, we observe that eighteen students define a function as being an “analytical expression”, twenty say that a function is “a relation/correspondence between two sets” and fifteen students define a function as being simultaneously an analytical expression and a relation/correspondence between sets. The next figure shows us an example of such answers (figure 4):



One function is an analytical expression that gives us a relation between two sets: that of the “arrived” and that of the “started”:



To each object corresponds one, and just one, image.

Figure 4 – The answer of the student A20 to Q_F.

This student still reveals us a structural perspective of function, where the object is represented by a letter (x), and the image by the symbol $f(x)$.

Though the teaching of the topic *Functions II* had been done with a strong emphasis on the graphical representation and the connections between the diverse representations of the function, twenty one students gave analytical examples of functions and only three presented a function by the graphical representation, one being a diagram (figure 5):

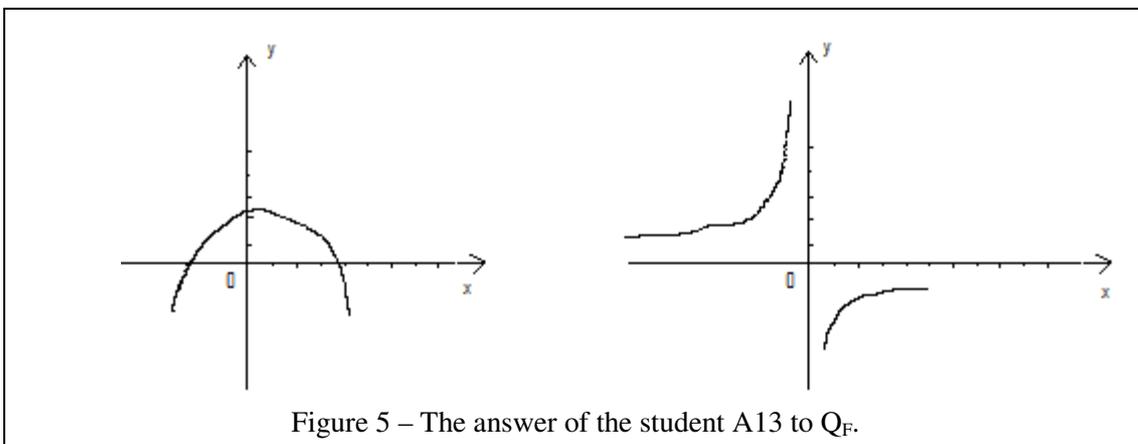


Figure 5 – The answer of the student A13 to Q_F.

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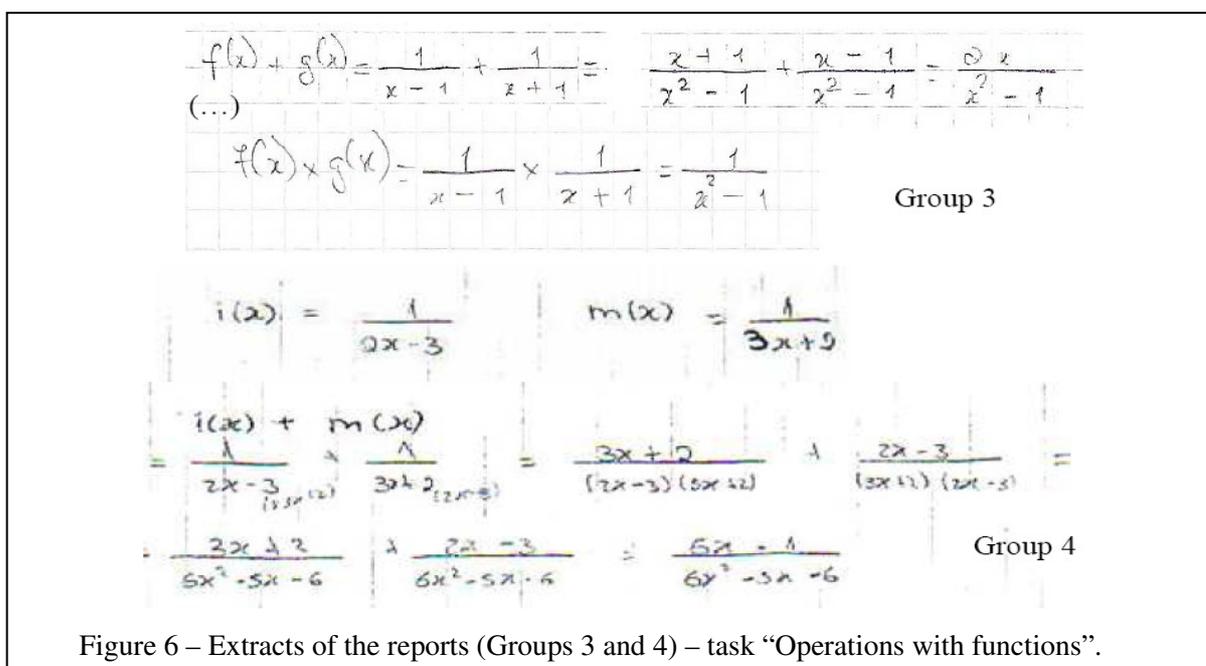
Although after the work emphasizing the connection between the different representations of a function, the students felt much more comfortable with the perspective of identifying, correctly, a function by its analytical representation, such as: “ $f(x)=3x^2+1/x$; and $g(x)=3x^3+2x^2+x+1$ (student A1)”.

Other aspects of algebra

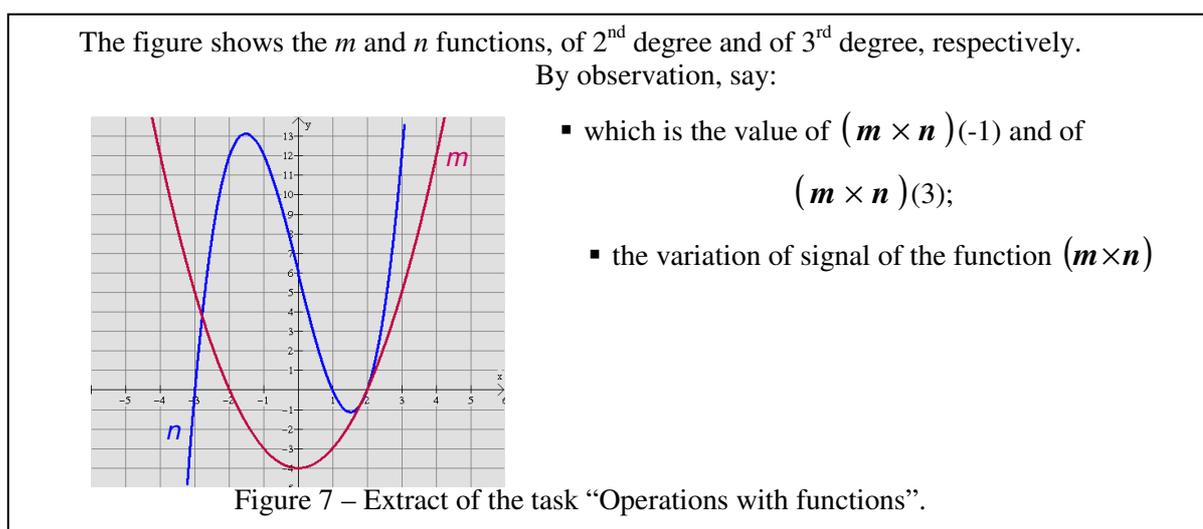
The friendship that students made with the variables influenced their performance in other tasks throughout the study. In the first part of the task “Operations with functions”, two functions

were given, expressed analytically, $f(x) = \frac{1}{x-1}$ and $g(x) = \frac{1}{x+1}$, and the students were

asked to identify the analytical expressions of the functions $(f+g)$, $(f-g)$ and $(f \times g)$. Afterwards, the students were invited to choose other pairs of functions, and to operate with them to find addition, difference and product functions and the relationship between the ranges of the original functions and the new ones. The students found the analytical expressions of the functions $(f+g)$, $(f-g)$ and $(f \times g)$ by the addition, difference and product, respectively, of the functions that were supplied to them or those that they had chosen. They carried through with relative effectiveness the necessary calculations (figure 6).



Nevertheless, the majority of the students had difficulty to define a strategy to solve the next task (figure 7).



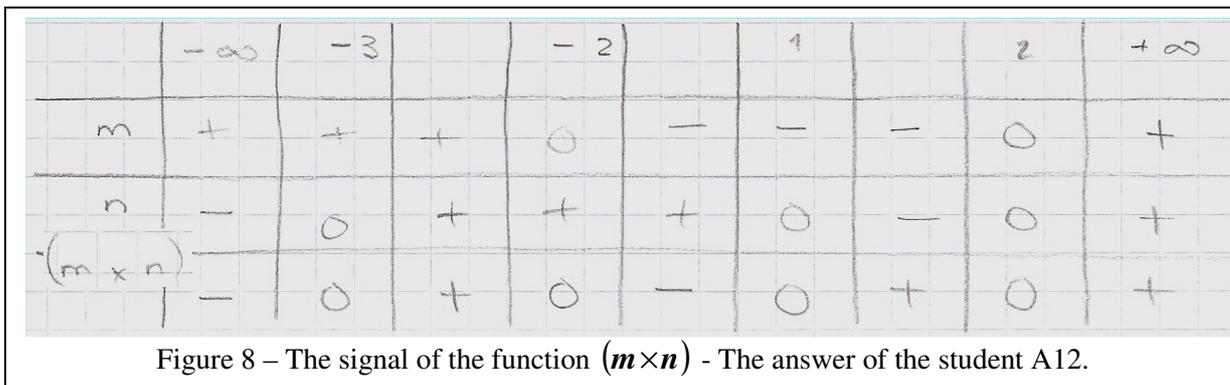
In fact, when students were faced with the inexistence of the analytical expression of the functions m and n , they did not collect the information supplied in graphical terms, and they didn't know what to do to determine the values of $(m \times n)(-1)$ and of $(m \times n)(3)$. The students' dialogue with the teacher was fundamental for them to understand that the required information

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was, now, presented graphically – and they were able to write “ $(m \times n)(-1) = m(-1) \times n(-1) = -3 \times 12 = -36$ and $(m \times n)(3) = m(3) \times n(3) = 5 \times 11 = 55$ ”.

In this case, the students’ confidence of a learned analytical routine was an obstacle for them to search the two concreteness of the variable x . One student wrote (student A5): “[in the previous task] it was easier, because we had x and the functions were given, and we knew already what to do...”.

In spite of the initial difficulties, the students’ exploration of the variation of signal of the function $(m \times n)$ shows us that they were able to understand, in a general way, what to search for in the functions m and n and to recall their previous knowledge in order to organize the answers (figure 8).



Resolutions and original strategies of the students

Throughout this study the students were involved in their own explorations, following their intuition. In the task “Transformations of rational functions” it was asked of the students to explore the influence of the parameters b , d and a , respectively, using the graphic calculator, in the families of functions of the type $y = \frac{b}{x}$ ($b \neq 0$), $y = \frac{1}{x+d}$ and $y = a + \frac{1}{x}$. Two students (A4 and A7) devoted some time to explore curiosities with the functions of the type $y = \frac{b}{x}$. In their report, written in the class, they wrote: “We thought that the “points of curvature” appeared in the same coordinates; we thought that the “points of curvature” belonged to the straight line $y = x$ ” (figure 9):

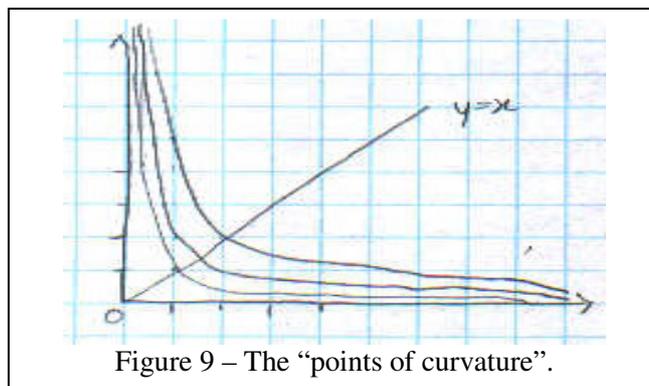


Figure 9 – The “points of curvature”.

Throughout the exploration of this conjecture, the students started to call the “points of curvature” as “points of change”. Later, this denomination was used in the whole class (in the discussion). Their reasoning was supported by the sketch of the graphics (figure 9). The symmetry of the hyperbole’s branches was analysed by the students using their intuition and they expressed this using correct terminology identifying it analytically ($y=x$). However, they refuted their formulated conjecture because they used an incorrect reasoning based on the fact that “point of change” of the function $y = \frac{1}{x}$ appears at (1, 1), so they thought that in the function $y = \frac{2}{x}$

the “point of change” would be at (2, 2), and in the function $y = \frac{3}{x}$ it will be at (3, 3), and so

on.

In the discussion class, where the conclusions were validated, this conjecture was analysed and the situation was explored using the table of values of the graphic calculator. Because it was not easy to identify the coordinates of the “point of change”, the situation was analysed analytically. The teacher/researcher suggested finding coordinates through a system of equations ($y = x$ and $y = \frac{b}{x}$, $b > 0$) and it was possible to conclude that the “points of change” are the points

whose coordinates are the form $V_1 (\sqrt{b}; \sqrt{b})$ and $V_2 (-\sqrt{b}; -\sqrt{b})$.

The collective negotiation of meanings allowed the students to discuss their ideas and to clarify and share opinions as to the personal understanding of the mathematical concepts. In fact, one of these two students explained how they arrived at calling these points as “points of change”. To the student, the “points of change” change the relation between the objects and the images. To the left of them, the objects are smaller than the images and after the objects are larger [$b > 0$]. This collective discussion also presented good opportunities for the students to express their difficulties and it was important to identify some misconceptions in the students’ learning. In fact, by searching analytically for the “points of change” [now with $b < 0$], the students’ interventions evidence one limited vision of the concept of variable and its number value. The students easily identified the symmetry axis of the hyperboles of the form $y = \frac{b}{x}$, $b < 0$, as the bisector of the even quadrants and with the straight line that contains this bisector, yet they said that the ana-

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lytical expression was $y = -x$. However, in the resolution of the correspondent system it was not easy to convince the students that the equation $x^2 = -b$ is possible and the solution is $x = \sqrt{-b} \vee x = -\sqrt{-b}$. Their reaction was as follows: “But it is not possible! There are no roots of negative numbers!” It was necessary to use many examples so that they could see that the symmetric of a negative number is a positive one.

Conclusion

Initially the definition of function was been memorized by most of the students, and they were not able to connect the words they write, such as “to a one object corresponds one and only one image”, with the graphical representation of a function. At the end of the study, most of the students associated the concept of function to an analytical expression (formula) and to a special relation, as Chazan *et al.* (2003) say. In general, the students connected the concept of function to an analytical expression which was fundamental to the graphing process – the teaching developed with the graphical representation of a function was not enough to the students give the necessary importance to the graphical representation; it continues not being the first association students made of a function.

Students’ capacity to operate algebraically with the analytical expressions, for example, adding and multiplying two functions, was not sufficient for the search of two concretizations of the variable with the graphic representation. The analytical expressions of each one of the two functions were necessary to solve the initial question. Clearly these students associated the concept of function with the analytical representation and, as Sajka (2003) says, the students’ capacity to manipulate the symbols, and to operate with them, is not sufficient for their structural understanding of a function.

This study clearly shows the benefit that students gain regarding symbol sense, as a direct consequence of the explorations they carried out with the graphical representations of functions and the study of its properties. The collective discussion of a misconception (“square root of $-b$ has no meaning”) allowed the students to extend the interpretation of letters as unknowns.

This study reinforces the importance of the research on the mathematical theme of functions, according to several authors (Arcavi, 2005; Ponte, 2005; Sajka, 2003; Sfard, 1991; Rojano, 2002). Despite mathematical notation ambiguity, the students’ mathematical activity, using exploratory and investigative mathematical tasks, in a classroom context with good interaction between the students and between them and the teacher, allowed for meaningful learning on functions to occur and, also, to identify the difficulties that the students faced.

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